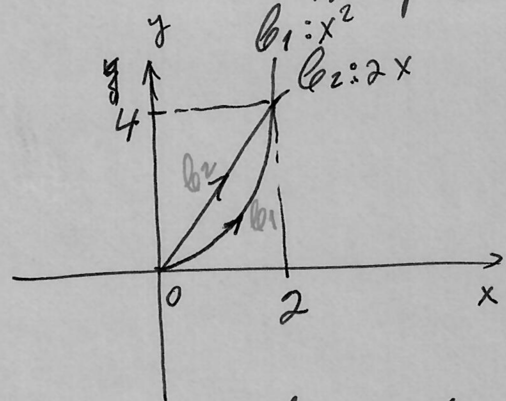
16.3 The Fundamental Theorem of Line Integrals

Example 1.– Consider the curves $\mathcal{C}_1$ and $\mathcal{C}_2$ in the plane given by $y = 2x$, $\mathcal{C}_2$; $y = x^2$, $\mathcal{C}_1$; $x \in [0,2]$



Compute the line integral of $\vec{F}(x,y) = \left[(x+2)y,\ 2x+y\right]$ over $\mathcal{C}_1$ and $\mathcal{C}_2$

$\mathcal{C}_1 : \vec{r}_1(t) = (t, t^2), \quad t \in [0,2]$

$\mathcal{C}_2 : \vec{r}_2(t) = (t, 2t), \quad t \in [0,2].$

Along $\mathcal{C}_1$:
$$\int_0^2 \vec{F}(\vec{r}_1(t)) \cdot \vec{r}_1{}'(t)\, dt =$$
$$= \int_0^2 \left((t+2)t^2,\ 2t+t^2\right) \cdot (1, 2t)\, dt =$$
$$= \int_0^2 \left[t^3 + 2t^2 + 4t^2 + 2t^3\right] dt = \int_0^2 \left[3t^3 + 6t^2\right] dt =$$
$$= 28.$$

Along $\mathcal{C}_2$:
$$\int_0^2 \vec{F}(\vec{r}_2(t)) \cdot \vec{r}_2{}'(t)\, dt =$$
$$= \int_0^2 \left((t+2)2t,\ 2t+2t\right) \cdot (1, 2)\, dt = \int_0^2 \left[2t^2 + 4t + 8t\right] dt =$$
$$= \int_0^2 (2t^2 + 12t)\, dt = \frac{88}{3}$$
$$\simeq 29.3$$

Comment in the light of work.

Example 2.– Modify $\vec{F}$ in example 1 by

$$\vec{F}(x,y) = (2y, 2x+y).$$

and compute line integrals over same paths C_1 and C_2.

Along C_1: $\int_0^2 (2t^2, 2t+t^2)(1, 2t)\, dt =$

$$= \int_0^2 (2t^2 + 4t^2 + 2t^3)\, dt = \int_0^2 (6t^2 + 2t^3)\, dt = \underline{24}$$

Along C_2:

$$\int_0^2 (4t, 2t+2t) \cdot (1,2)\, dt = \int_0^2 (4t, 4t) \cdot (1,2)\, dt =$$

$$= \int_0^2 (4t + 8t)\, dt = \int_0^2 12t\, dt = \underline{\underline{24}}$$

Same work why

Line integrals are path independent!

$$I = \int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \int_a^b \langle P(), Q(), R() \rangle \cdot \vec{r}\,'(t)\, dt$$

$$= \int_a^b \langle P(), Q(), R() \rangle \cdot \langle x'(t), y'(t), z'(t) \rangle \, dt$$

$$= \int_a^b P(x(t), y(t), z(t))\, x'(t)\, dt + \int_a^b Q(x(t), y(t), z(t))\, y'(t)\, dt$$

$$+ \int_a^b R(x(t), y(t), z(t))\, z'(t)\, dt$$

$$= \int_{\mathcal{C}} P(x,y,z)\, dx + \int_{\mathcal{C}} Q(x,y,z)\, dy + \int_{\mathcal{C}} R(x,y,z)\, dz$$

$$= \int_{\mathcal{C}} P dx + Q dy + R dz .$$

16.3 The fundamental Theorem of Line Integrals.

Theorem.-

- $\mathcal{C}$ is a Smooth curve given by

$$\mathcal{C}: \vec{r}(t) = \langle x(t), y(t), z(t) \rangle, \qquad a \leq t \leq b.$$

- f is a differentiable function

- $\nabla f(x,y,z)$ is continuous on $\mathcal{C}$.

Then,

$$\int_{\mathcal{C}} \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)).$$

Previous Consideration:

$$\int_a^b g'(t)\,dt = g(b) - g(a) \quad \text{F.T.C.}, \quad \text{If } g(t) \overset{(1)}{=} f(\vec{r}(t)) \overset{(2)}{=} f(x(t), y(t), z(t))$$

Then

$$\int_a^b \frac{d}{dt}\left[f(\vec{r}(t))\right] dt = \int_a^b g'(t)\,dt =$$

$$= g(b) - g(a) \overset{(1)}{=} f(\vec{r}(b)) - f(\vec{r}(a))$$

$$\overset{(2)}{=} f(x(b), y(b), z(b)) - f(x(a), y(a), z(a))$$

<u>Proof</u>.- (Thm1: Integration of Conservative Vector fields.)

First consider that

$$\frac{d}{dt} f(\vec{r}(t)) = \frac{d}{dt} f(x(t), y(t), z(t))$$

$$\overset{\text{Chain rule}}{=} \frac{\partial f}{\partial x}(\vec{r}(t))\, x'(t) + \frac{\partial f}{\partial y}(\vec{r}(t))\, y'(t) + \frac{\partial f}{\partial z}(\vec{r}(t))\, z'(t)$$

$$= \nabla f(\vec{r}(t)) \cdot \langle x'(t), y'(t), z'(t) \rangle = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t).$$

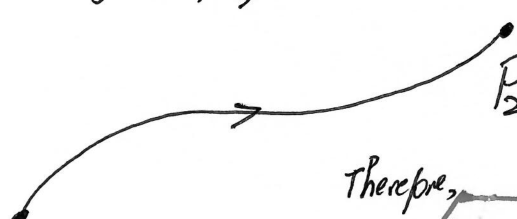
Therefore,

$$\int_{\mathscr{C}} \nabla f \cdot d\vec{r} = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_a^b \frac{d}{dt} f(\vec{r}(t))\, dt \overset{\substack{\text{Fund. Thm} \\ \text{of Calculus}}}{=} f(\vec{r}(b)) - f(\vec{r}(a))$$

$$= f(x(b), y(b), z(b)) - f(x(a), y(a), z(a))$$

$$= f(x_2, y_2, z_2) - f(x_1, y_1, z_1) = f(P_2) - f(P_1).$$



$P_2\,(x_2, y_2, z_2) = \vec{r}(b)$

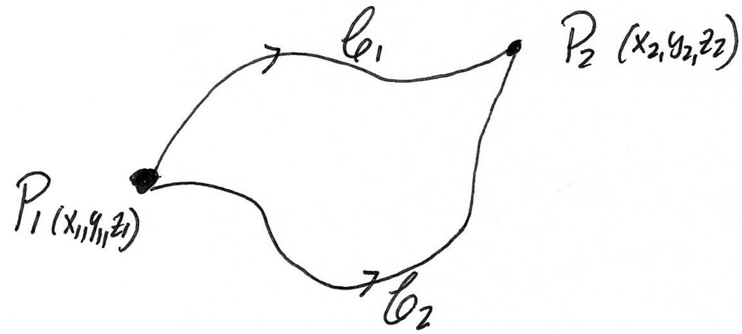
$P_1\,(x_1, y_1, z_1) = \vec{r}(a)$

Therefore,

$$\boxed{\int \nabla f \cdot d\vec{r} = f(P_2) - f(P_1)}$$

Corollary 1.- (Independence of path)

- $\mathcal{C}_1$ is a smooth curve given by $\vec{r}_1(t)$,
  $a \le t \le b$ and $\vec{r}_1(a) = \langle x_1, y_1, z_1 \rangle$, $\vec{r}_1(b) = \langle x_2, y_2, z_2 \rangle$
- $\mathcal{C}_2$ is a smooth curve given by $\vec{r}_2(u)$
  $c \le u \le d$ and $\vec{r}_2(c) = \langle x_1, y_1, z_1 \rangle$, $\vec{r}_2(d) = \langle x_2, y_2, z_2 \rangle$



- f is defined in a domain D containing $\mathcal{C}_1$ and $\mathcal{C}_2$
- f is differentiable and ∇f is continuous on $\mathcal{C}_1$ and $\mathcal{C}_2$

Then,

$$\boxed{\int_{\mathcal{C}_1} \nabla f \cdot d\vec{r} = \int_{\mathcal{C}_2} \nabla f \cdot d\vec{r}}$$

Proof.-

$$\int_{\mathcal{C}_1} \nabla f \cdot d\vec{r} = f(x_2, y_2, z_2) - f(x_1, y_1, z_1) = \int_{\mathcal{C}_2} \nabla f \cdot d\vec{r}$$

<u>Corollary 2</u> (Line Integrals of Conservative Vector fields are independent of the path)

- F Continuous on $D \subseteq \mathbb{R}^3$
- $\mathcal{C}_1$ and $\mathcal{C}_2$ are Smooth curves contained in D with Same initial and terminal points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, respectively.
- F is a conservative Vector field on the domain D. It means, there is f such that $\nabla f = \vec{F}$

Then,

$$\boxed{\int_{\mathcal{C}_1} \vec{F}.d\hat{r} = \int_{\mathcal{C}_2} \vec{F}.d\hat{r}}.$$

<u>Proof.</u>

$$\int_{\mathcal{C}_1} \vec{F}.d\hat{r} = \int_{\mathcal{C}_1} \nabla f.d\hat{r} = f(x_2, y_2, z_2) - f(x_1, y_1, z_1)$$

$$= \int_{\mathcal{C}_2} \nabla f.d\hat{r} = \int_{\mathcal{C}_2} \vec{F}.d\hat{r}.$$

Back to our example,

$$F(x,y) = \langle 2y, 2x+y \rangle.$$

$C_1: \vec{r}_1(t) = \langle t, t^2 \rangle, \quad 0 \le t \le 2.$

$C_2: \vec{r}_2(t) = \langle t, 2t \rangle, \quad 0 \le t \le 2$

Since $\displaystyle\int_{C_1} \vec{F}.d\vec{r} = 24 = \int_{C_2} \vec{F}.d\vec{r}$ (Review previous comp.)

We want to investigate if $\vec{F}$ is a conservative Vector field.

For proving this, we need to find a scalar function $f(x,y)$ (called potential of $\vec{F}$) such that

$$\nabla f(x,y) = \vec{F}(x,y) = \langle 2y, 2x+y \rangle \qquad (*)$$

<u>Procedure</u>: For (*) to verify, we need

① $f_x(x,y) = 2y$ leads $\int dx$: $f(x,y) = 2xy + C(y)$ ③

② $f_y(x,y) = 2x+y$

③ leads to $f_y(x,y) = 2x + C'(y)$

Combining with ②, $\cancel{2x} + C'(y) = f_y(x,y) = \cancel{2x} + y$

Then $C'(y) = y$ and $C(y) = \frac{y^2}{2} + d$

Finally, $\boxed{f(x,y) = 2xy + \frac{y^2}{2},}$ $(d=0)$.

Clearly, $\quad \nabla f(x,y) = \langle \overset{f_x}{2y}, \overset{f_y}{2x+y} \rangle = \vec{F}(x,y)$

So $\vec{F}$ is a conservative vector field with potential function $\boxed{f(x,y) = 2xy + \frac{y^2}{2}}$

Therefore, to compute

$$\int_{C_1} \vec{F}.d\vec{r} \quad \text{and} \quad \int_{C_2} \vec{F}.d\vec{r}$$

We only need to evaluate f at the end points and Subtract.

In fact,

$$\int_{C_1} \vec{F}.d\vec{r} = f(2,4) - f(0,0) = 2(2)(4) + \frac{4^2}{2} = \underline{\underline{24}}$$

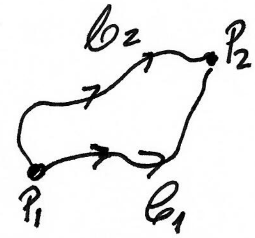
$$\int_{C_2} \vec{F}.d\vec{r} = f(2,4) - f(0,0) = \cdots = 24.$$

What is $\displaystyle\int_{C_3} \vec{F}.d\vec{r}$ for another curve C_3 (piecewise) between $P(0,0)$ and $P_2(2,4)$?

<u>Definition</u> (Independence of path).

- C_1 and C_2 are two piecewise smooth curves (paths) contained in a domain D.
- $\vec{F}$ is a continuous vector field with domain D.
- C_1 and C_2 have the same initial and terminal points

If
$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$$



for any two paths C_1 and C_2 in D
we say that the <u>line integral</u> $\int_C \vec{F}\cdot d\vec{r}$ <u>is independent of path</u>.

<u>Definition:</u> A curve C is closed if its terminal point coincides with its initial points.

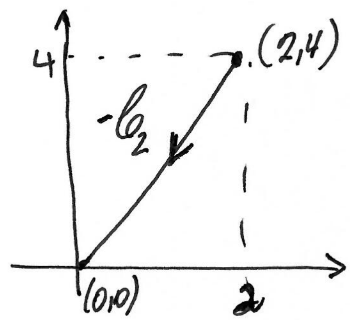
<u>Theorem 3.</u>- $\int_C F\cdot d\vec{r}$ is independent of path in D

if and only if
$$\int_C \vec{F}\cdot d\vec{r} = 0$$
for every closed path C in D.

Before proving Thm 3, let's compute

$$I = \int_{-\mathcal{C}_2} \langle 2y, 2x+y \rangle \cdot d\vec{r}$$

where

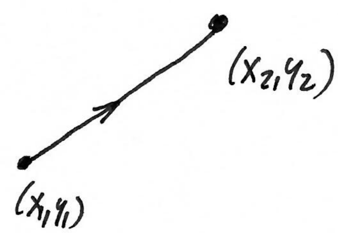


$-\mathcal{C}_2$ consists of the same path of $\mathcal{C}_2$ but it is travelled in the opposite direction.

How can we define $-\mathcal{C}_2$?

General idea: Find parametric curve $\mathcal{C}$ given by a straight segment between (x_1, y_1) and (x_2, y_2).

$$\mathcal{C}: \ (1-t)(x_1, y_1) + t(x_2, y_2), \quad 0 \le t \le 1$$



So, in our example

$$-\mathcal{C}_2: \left. \begin{array}{l} (1-t)(2,4) + t(0,0) \\ 0 \le t \le 1 \end{array} \right\} = \langle 2-2t, 4-4t \rangle = \vec{r}(t)$$

Thus,

$$I = \int_0^1 \langle 2(4-4t), 2(2-2t) + 4-4t \rangle \cdot \underbrace{\langle -2, 4 \rangle}_{= \vec{r}\,'(t)} dt$$

$$= \int_0^1 \langle 8-8t, 8-8t \rangle \cdot \langle -2, 4 \rangle dt = \int_0^1 (48t - 48) dt = \underline{\underline{-24}}.$$

Thus, for our particular Vector field

$F(x,y) = \langle 2y, 2x+y \rangle$,

we have proved that

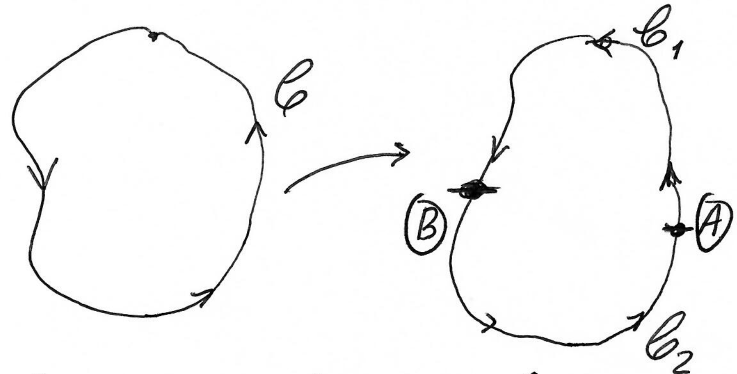
$$\int_{-\mathcal{C}_2} \vec{F} \cdot d\vec{r} = -24 = -\int_{\mathcal{C}_2} \vec{F} \cdot d\vec{r}$$

This is true in general for any piecewise Curve $\mathcal{C}$ and a Vector field $\vec{F}$.

$$\int_{-\mathcal{C}} F \cdot d\vec{r} = -\int_{\mathcal{C}} \vec{F} \cdot d\vec{r}$$

Proof (Thm 3)

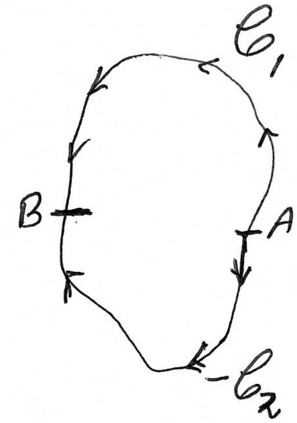
($\longrightarrow$) Assume $\int_{\mathcal{C}} \vec{F}\cdot d\vec{r}$ is independent of path and consider an arbitrary closed curve $\mathcal{C}$



$\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$: Ⓑ initial point of $\mathcal{C}_2$ is also a terminal point $\mathcal{C}_1$
Ⓐ initial point $\mathcal{C}_1$ is also a terminal point $\mathcal{C}_2$.

Then,

$$\int_{\mathcal{C}} \vec{F}\cdot d\vec{r} = \int_{\mathcal{C}_1} \vec{F}\cdot d\vec{r} + \int_{\mathcal{C}_2} \vec{F}\cdot d\vec{r}$$



$$\underset{\text{Previous results}}{=} \int_{\mathcal{C}_1} \vec{F}\cdot d\vec{r} - \int_{-\mathcal{C}_2} \vec{F}\cdot d\vec{r} = 0$$

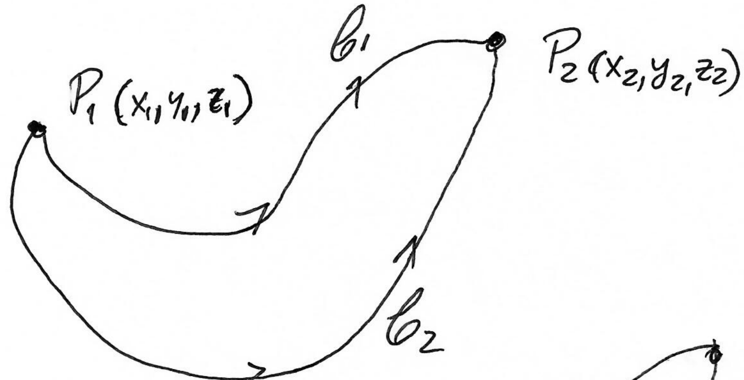
Since $\mathcal{C}_1$ and $-\mathcal{C}_2$ have same initial and terminal points.

($\Leftarrow$) Assume

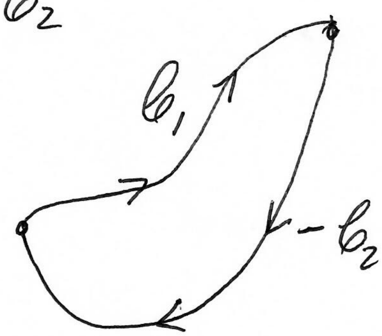
$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = 0, \quad \text{for any closed curve } \mathcal{C}$$

piecewise smooth.

and consider two different paths $\mathcal{C}_1$ and $\mathcal{C}_2$ between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$. (arbitrary points) on $\mathcal{C}$.



We can form a closed curve



$$\mathcal{C} = \mathcal{C}_1 \cup (-\mathcal{C}_2)$$

Thus,

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \int_{\mathcal{C}_1} \vec{F} \cdot d\vec{r} + \int_{-\mathcal{C}_2} \vec{F} \cdot d\vec{r} = 0$$

$$\Rightarrow \int_{\mathcal{C}_1} F \cdot dr = -\int_{-\mathcal{C}_2} \vec{F} \cdot d\vec{r} = \int_{\mathcal{C}_2} \vec{F} \cdot d\vec{r}$$

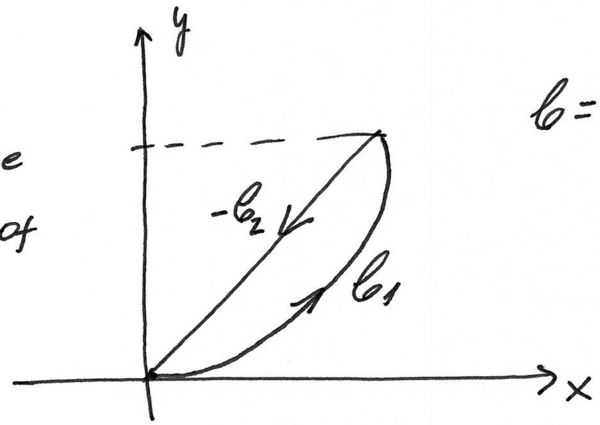
Application to our vector field

$$F(x,y) = \langle 2y, 2x+y \rangle$$

and $\quad C_1 : r_1(t) = \langle t, t^2 \rangle, \quad 0 \le t \le 2$

$\qquad\quad C_2 : r_2(t) = \langle t, 2t \rangle, \quad 0 \le t \le 2$

We know
$\vec{F}$ is Conservative
and as consequence
it is independent of
path.



$C = C_1 \cup (-C_2)$ is closed

And
$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot dr + \int_{-C_2} \vec{F} \cdot d\vec{r} = 24 - 24 = 0.$$

So $\int_C \vec{F} \cdot d\vec{r}$. zero along the closed curve $C = C_1 \cup (-C_2)$.
shown in the graph above.

Construction of a Potential for a Conservative Vector Field.

We already noticed that

for $F_1(x,y) = \langle 2y, 2x+y \rangle$ the scalar function

$f(x,y) = 2xy + \frac{y^2}{2}$ is such that

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$$

$$= \langle 2y, 2x+y \rangle = F_1(x,y)$$

So $f(x,y) = 2xy + \frac{y^2}{2}$ is a potential function for the vector field $F_1(x,y)$.

The following steps lead to obtain f:

1) Integrate w.r.t. x

$$f_x(x,y) = 2y$$

So $f(x,y) = 2xy + C(y)$ constant depends on other variables different than x.

2) Take derivative of $f(x,y)$ w.r.t. y equal Q, and integrate w.r.t. y

$$f_y(x,y) = 2x + C'(y) = 2x + y$$

then $C'(y) = y$. and $C(y) = \frac{y^2}{2} + d$

or $\boxed{f(x,y) = 2xy + \frac{y^2}{2} + d}$ $=0$

Only vector fields that are independent of path are conservative.

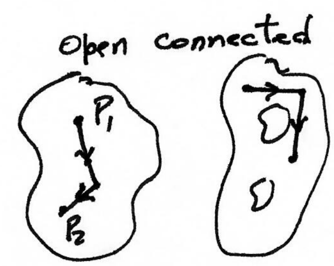
Theorem 4. (Reciprocal of Corollary 2 in p. 3)

- $\vec{F}$ is a continuous vector field on D.
- D is an open connected region.

If $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in D,

then $\vec{F}$ is a conservative vector field on D.

Proof. (In book).



Theorem 5

- $\vec{F}$ is a conservative vector field; $F(x,y) = \langle P(x,y), Q(x,y) \rangle$
- P and Q have continuous first-order partial derivatives on D.

Then,

$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \text{on } D.}$$

Proof: If $\vec{F}$ is conservative, then there exists $f(x,y)$ such that

$$\vec{F}(x,y) = \nabla f(x,y), \quad \text{on } D.$$

$$\langle P(x,y), Q(x,y) \rangle = \langle f_x(x,y), f_y(x,y) \rangle$$

This is equivalent to say

$$f_x(x,y) = P(x,y) \quad \text{and} \quad f_y(x,y) = Q(x,y)$$

Using Clairaut's theorem

$$P_y(x,y) = f_{xy}(x,y) = f_{yx}(x,y) = Q_x(x,y).$$

or

$$\frac{\partial P}{\partial y}(x,y) = \frac{\partial Q}{\partial x}(x,y).$$

Can we use this condition to determine if a vector field is conservative?

Notice that

$$F_1(x,y) = \langle 2y, 2x+y \rangle = \langle P(x,y), Q(x,y) \rangle$$

Satisfies

$$\frac{\partial P}{\partial y} = 2 = \frac{\partial Q}{\partial x}$$

but

$$F_2(x,y) = \langle (x+2)y, 2x+y \rangle = \langle P(x,y), Q(x,y) \rangle$$

$$\frac{\partial P}{\partial y} = x+2 \neq 2 = \frac{\partial Q}{\partial x}$$

Under certain conditions on the domain D, it can be proved that the above condition is also sufficient for a vector field $\vec{F}$ to be conservative.

Theorem 6

- $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ is a vector field. on D
- D is an open simply-connected region D.
- P and Q have continuous first order partial derivatives.

$$\boxed{\text{If } \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \text{ then } \vec{F} \text{ is Conservative}}$$

Sect. 16.3

#18) $\boxed{\vec{F}(x,y,z) = \langle \sin y,\ x\cos y + \cos z,\ -y\sin z \rangle}$

Assuming $\vec{F}$ is conservative, let's find its potential f.

$$f_x(x,y,z) = \sin y \overset{\int dx}{\Rightarrow} f(x,y,z) = x\sin y + C(y,z)$$

So $f_y(x,y,z) = x\cos y + \frac{\partial C}{\partial y}(y,z) = x\cos y + \cos z$

$$\Rightarrow \quad \frac{\partial C}{\partial y}(y,z) = \cos z \overset{\int dy}{\Rightarrow} C(y,z) = y\cos z + d(z)$$

$$\Rightarrow \quad f(x,y,z) = x\sin y + y\cos z + d(z)$$

So $f_z(x,y,z) = -y\sin z + d'(z) = -y\sin z$

$$\Rightarrow \quad d'(z) = 0 \Rightarrow d(z) = K \text{ (const.)}$$

Then, $\boxed{f(x,y,z) = x\sin y + y\cos z + K}$

is a potential function for $\vec{F}$.

How can we know that a vector field $\vec{F}$ defined in $D \subset \mathbb{R}^3$ is conservative?

Answer: Theorem 4 Section 16.5.